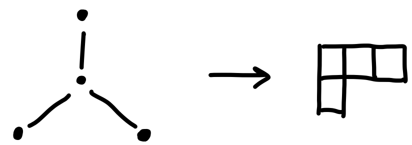
The Greene-Kleitman Correspondence
Jordan Forms & Flag Varieties.

① GK- correspondence

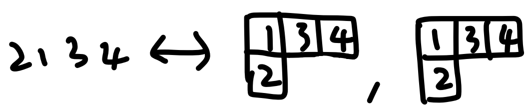
Greene MR0398912

Greene-Kleitman MR0398844



Fomin MR514781

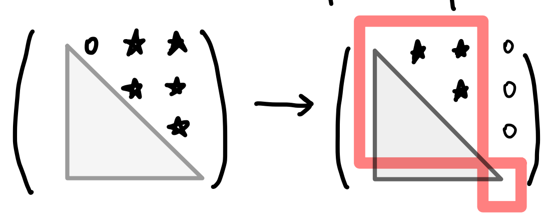
② Robinson- Schensted algorithm



Gansner MR0634367

Saks MR0578324

③ Jordan Form of Nilpotent matrices



Steinberg MR0929778

④ Relative position of pairs of flags

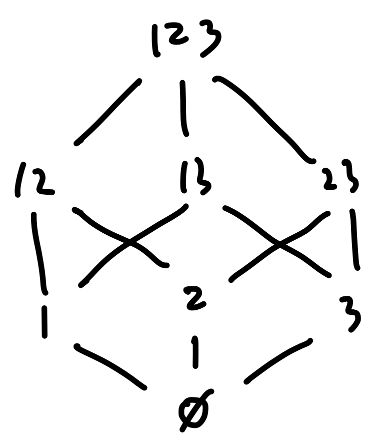
☆ Survey by Britz-Fomin: arxiv:9912126.

§ 0 Background on Posets & Dilworth's theorem

<u>Def</u> A partially ordered set (poset) is a set with an binary relation $\leq$ that is anti-sym, reflexiv, & transitive.

<u>Def</u> The Hasse diagram of a poset is a graph s.t. $\begin{matrix} b \\ | \\ a \end{matrix} \rightarrow a < b$

<u>Eg.</u> Boolean Lattice:

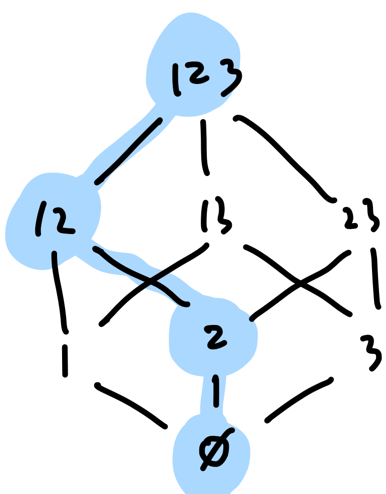


Some more definitions...

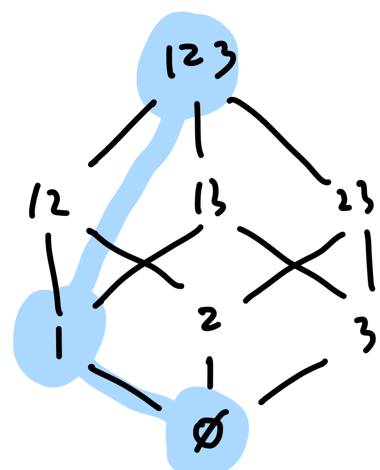
Def A chain is a totally ordered subset.

Def An antichain is a pair-wise incomparable subset.

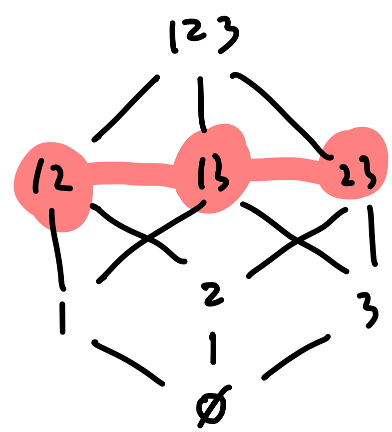
E.g.



Saturated chain



chain



antichain

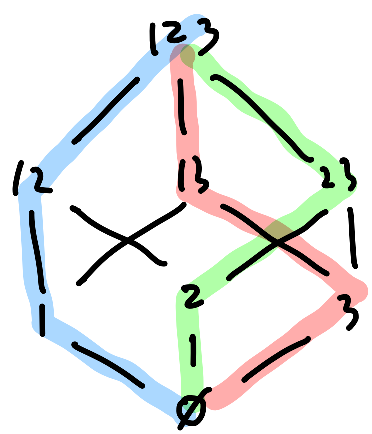
Dilworth's theorem

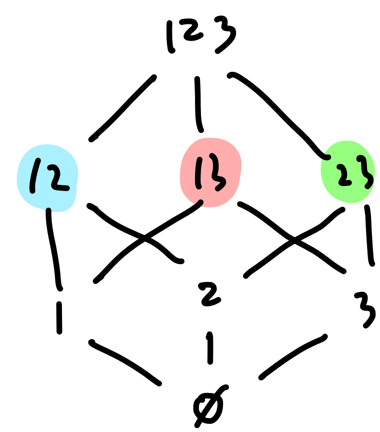
Thm (Dilworth 1950)

The minimum number of chains to cover a poset P

$=$ the size of maximal anti chain in P.

E.g





A Dual version

Thm (Dilworth 1950)

The minimum number of chains to cover a poset P

$=$ the size of maximal anti chain in P.

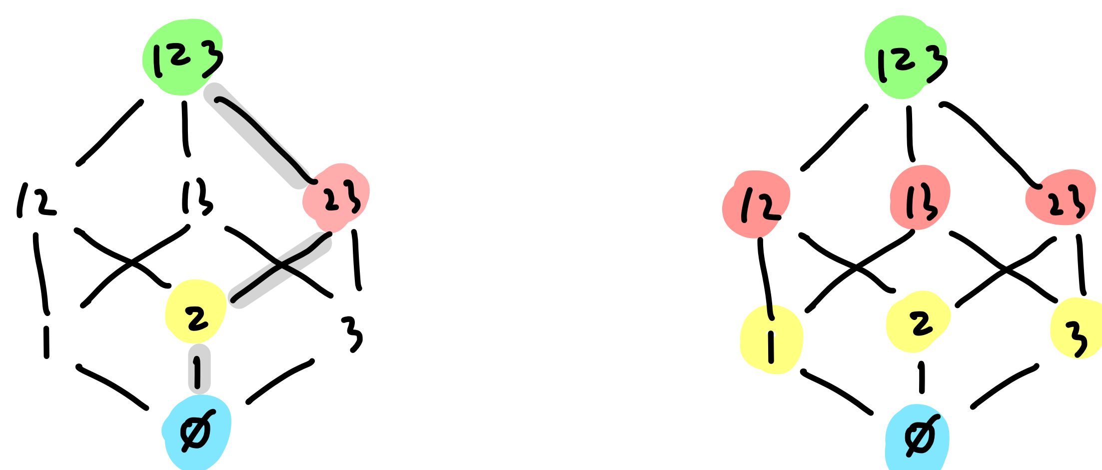
A Dual version

Thm (Mirsky)

The minimum number of antichains to cover a poset P

= the size of maximal chain in P.

E.g.



§1 The Greene-Kleitman Correspondence.

<u>Motivation</u> maximal size of a chain = min antichain cover,

what about maximal size of union of more chains?

<u>Def</u> A k-(anti)chain is a family of k (anti)chains.

<u>Def</u> For a poset P. define:

$$c_k := \text{maximal size of a } k\text{-chain of } P$$

$$a_k := \text{maximal size of a } k\text{-antichain of } P$$

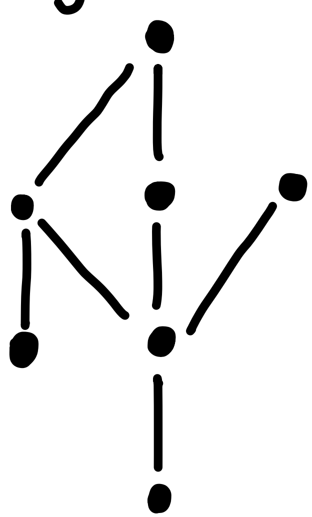
$\underline{\text{Def}}$ (GK-correspondence)

For a poset P, define

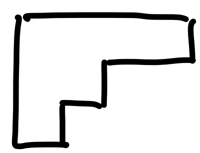
$$\lambda_1 = c_1 - c_0 \qquad \mu_1 = a_1 - a_0$$
$$\lambda_2 = c_2 - c_1 \qquad \mu_2 = a_2 - a_1$$
$$\lambda_3 = c_3 - c_2 \qquad \mu_3 = a_3 - a_2$$
$$\vdots \qquad\qquad \vdots$$

Then $\lambda(P) := (\lambda_1, \lambda_2 \cdots)$ and $\mu(P) := (\mu_1, \mu_2 \cdots)$ are weakly decreasing, and are both integer partitions of $\# P$.

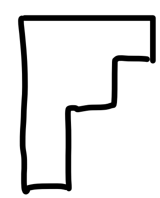
E.g.



$$\lambda_1 = 4 - 0 = 4$$
$$\lambda_2 = 6 - 4 = 2$$
$$\lambda_3 = 7 - 6 = 1$$
$$\lambda_4 = 7 - 7 = 0$$
$$\vdots$$



$$a_1 = 3 - 0 = 3$$
$$a_2 = 5 - 3 = 2$$
$$a_3 = 6 - 5 = 1$$
$$a_4 = 7 - 6 = 1$$
$$a_5 = 7 - 7 = 0$$
$$\vdots$$

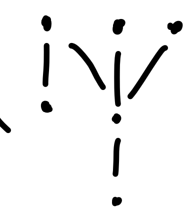
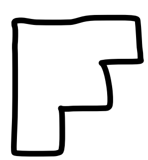
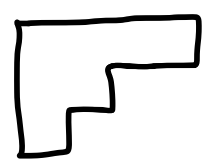


Theorem 1 (Greene, Greene-Kleitman)

$\lambda(P)$ and $\mu(P)$ are conjugate (dual) integer partitions of $\#P$.

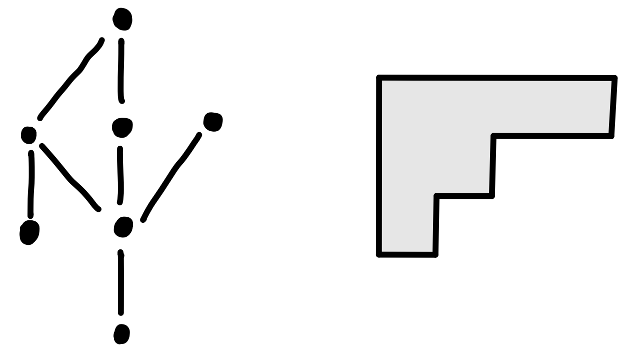
Theorem 2 (Fomin)

If a is a minimal or maximal element of P then $\lambda(P - \{a\}) \subset \lambda(P)$

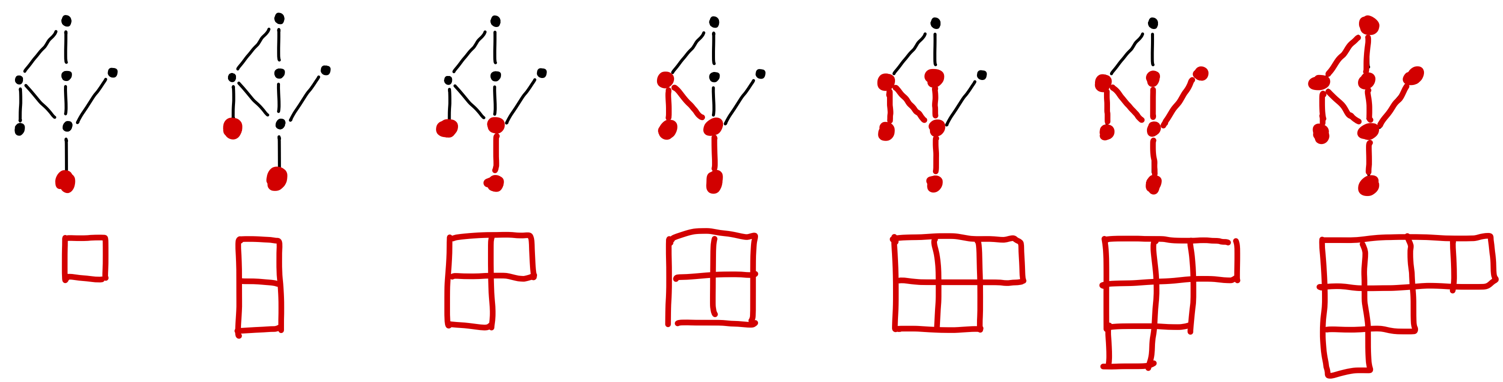
E.g. $\lambda\Big($  $\Big) =$  $\subset$ 

Some remarks.

Thm 1 (duality) generalize Dilworth's thm & Mirsky's thm simultaneously.



Thm 2 (monotonicity) means that we can calculate $\lambda(P)$ recursively by adding maximal elements.



§ 2 Tableaux Theory

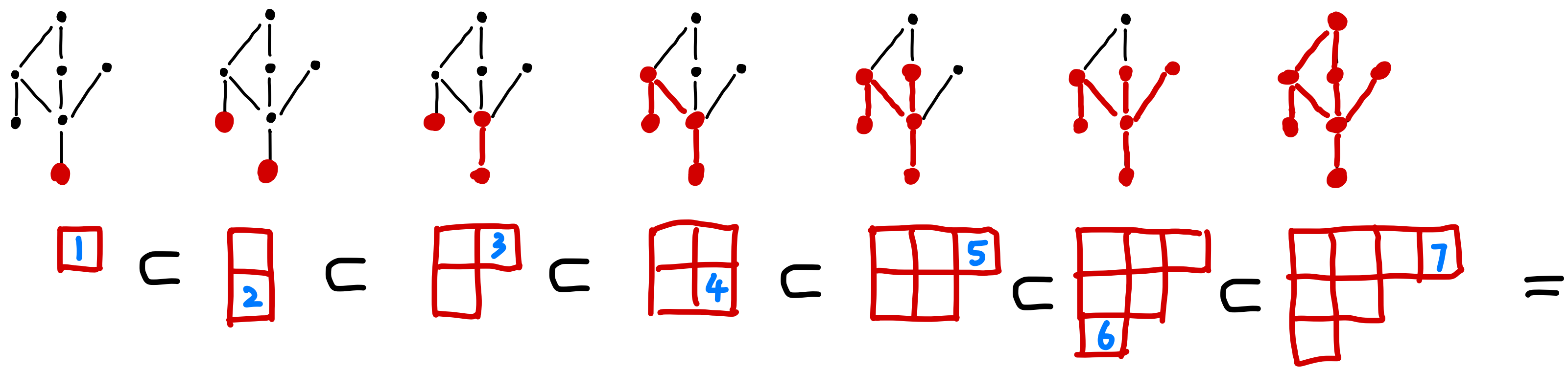
Def A Semi Standard Young tableau is a filling of a Young diagram by $[n]$ s.t. columns are increasing
rows are weakly increasing.

A Standard Young Tableau (SYT) is is a SSYT without repeated entries.

The underlying Young diagram is the shape of a tableau.

Eg.
$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \text{ is a SSYT} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \text{ is a SYT}$$

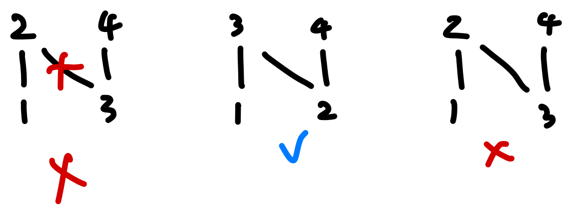
SYTs are flags of Young diagrams, so ...



$$= \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 4 & & \\ 6 & & & \end{matrix}$$

Def A *linear extension* of P is a numbering of P by $\{1, \cdots, \#P\}$ which preserves the partial order.

E.g.



Cor Thm 2 $\Rightarrow$ Every linear extension of P gives a SYT whose shape is $\lambda(P)$

Robinson-Schensted (RS) Algorithm.

<u>Def/Thm</u> Every permutation $w \in S_n$ is in bijection with a pair of SYT's $\{P(w), Q(w)\}$ of the same shape. defined via the RS insertion algorithm

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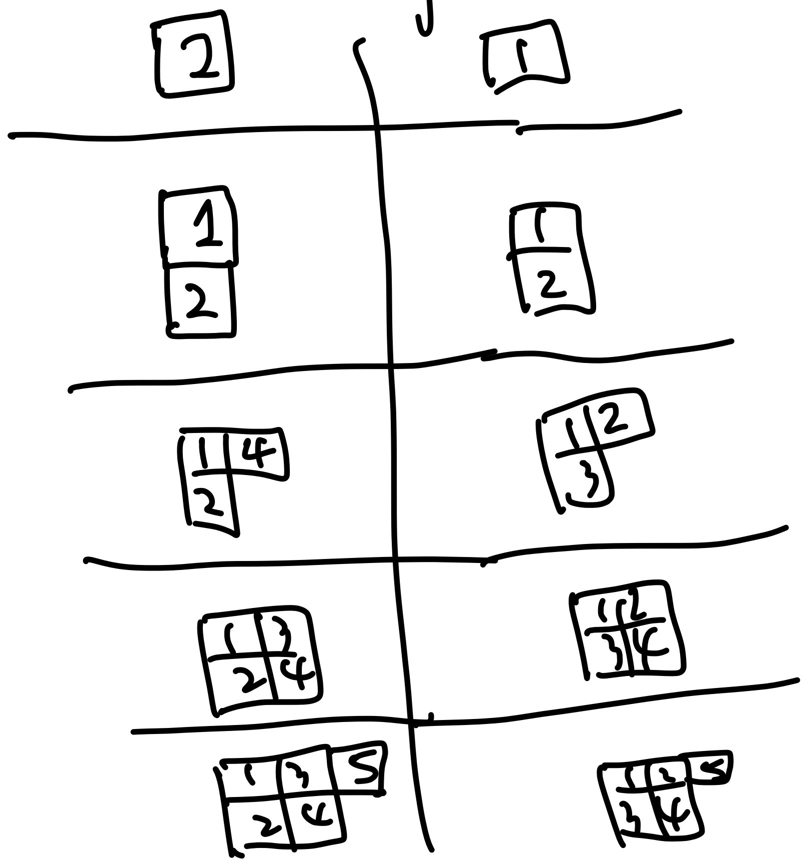

Fun Facts about RS ..

- length of 1st row / column of $P(w)$ is the length of longest increasing / decreasing subsequence.

- $$\sum_{\lambda \vdash n} \#\{T : \text{shape}(T) = \lambda\}^2 = n!$$

- $\{w : P(w) = T\}$
  $\{w : Q(w) = T\}$ are Kazhdan-Lusztig left / right / two-sided cells
  $\{w : \text{shape}\, P(w) = \lambda\}$

From GK to RS

Def There is a poset attached to each permutation w, called the inversion poset.

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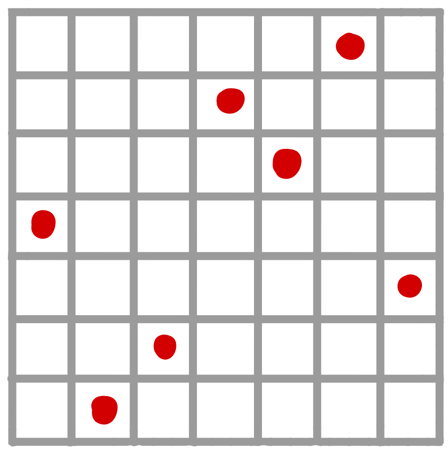

From GK to RS

Def There is a poset attached to each permutation w, called the inversion poset, denoted P_w.

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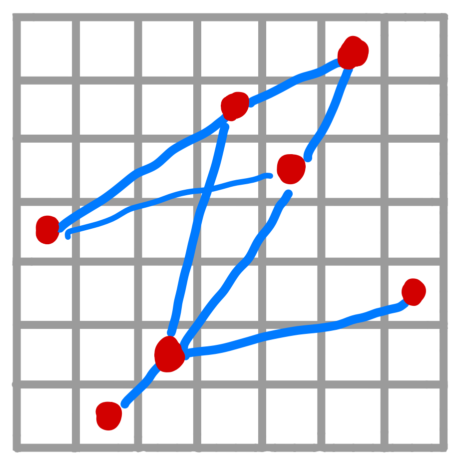
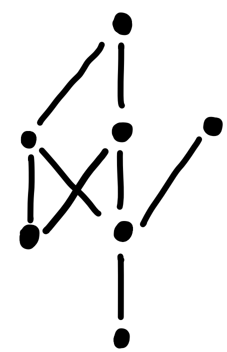
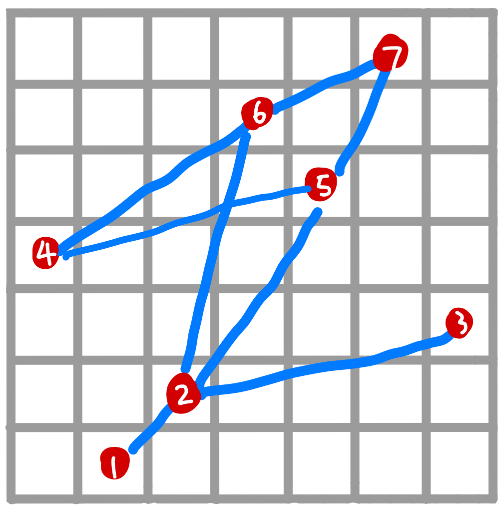




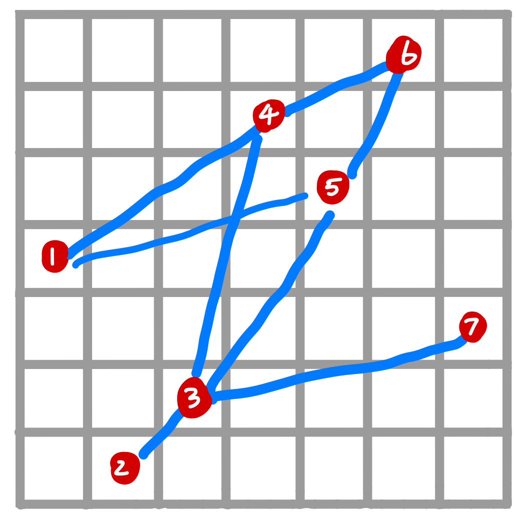
Thm (Greene, Fomin) by example

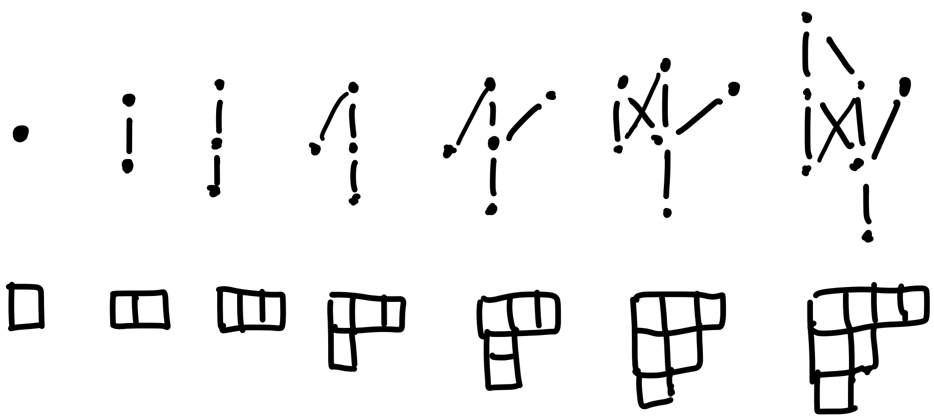
There are two natural linear extensions on P_w by row & column index.

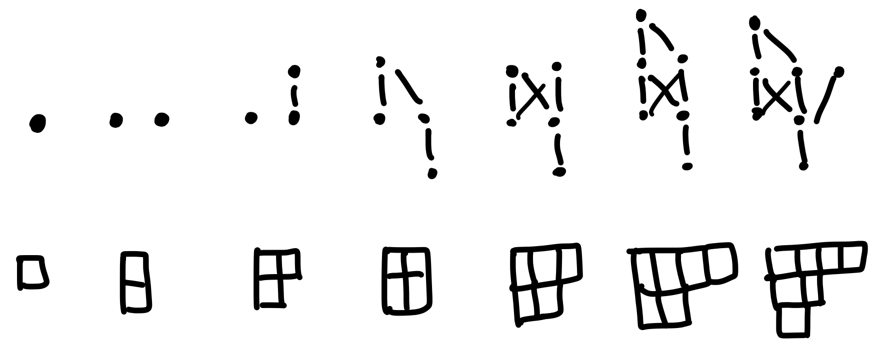
Row Index



Column Index







$$\begin{array}{llll} 1 & 2 & 3 & 7 \\ 4 & 6 & & \\ 5 & & & \end{array} = P(4762135)$$

$$\begin{array}{llll} 1 & 3 & 5 & 6 \\ 2 & 4 & & \\ 7 & & & \end{array} = Q(4762135)$$

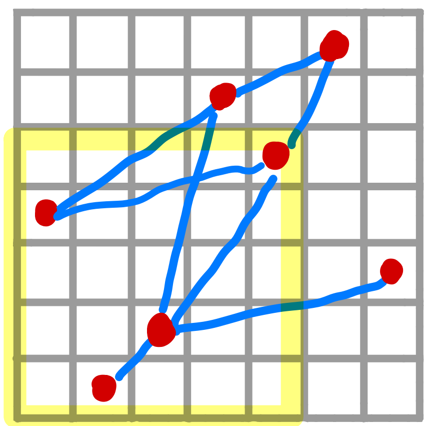
Another way to see this...

In each box of the $[n]\times[n]$ board, we put a subposet of P_w

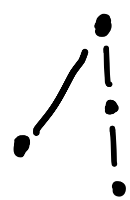
$$P_w^{(i,j)} = [i]\times[j] \cap P_w$$

And replace it with $\lambda_{ij} = \lambda\left(P_w^{(i,j)}\right)$

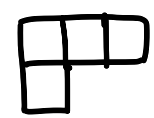
E.g.



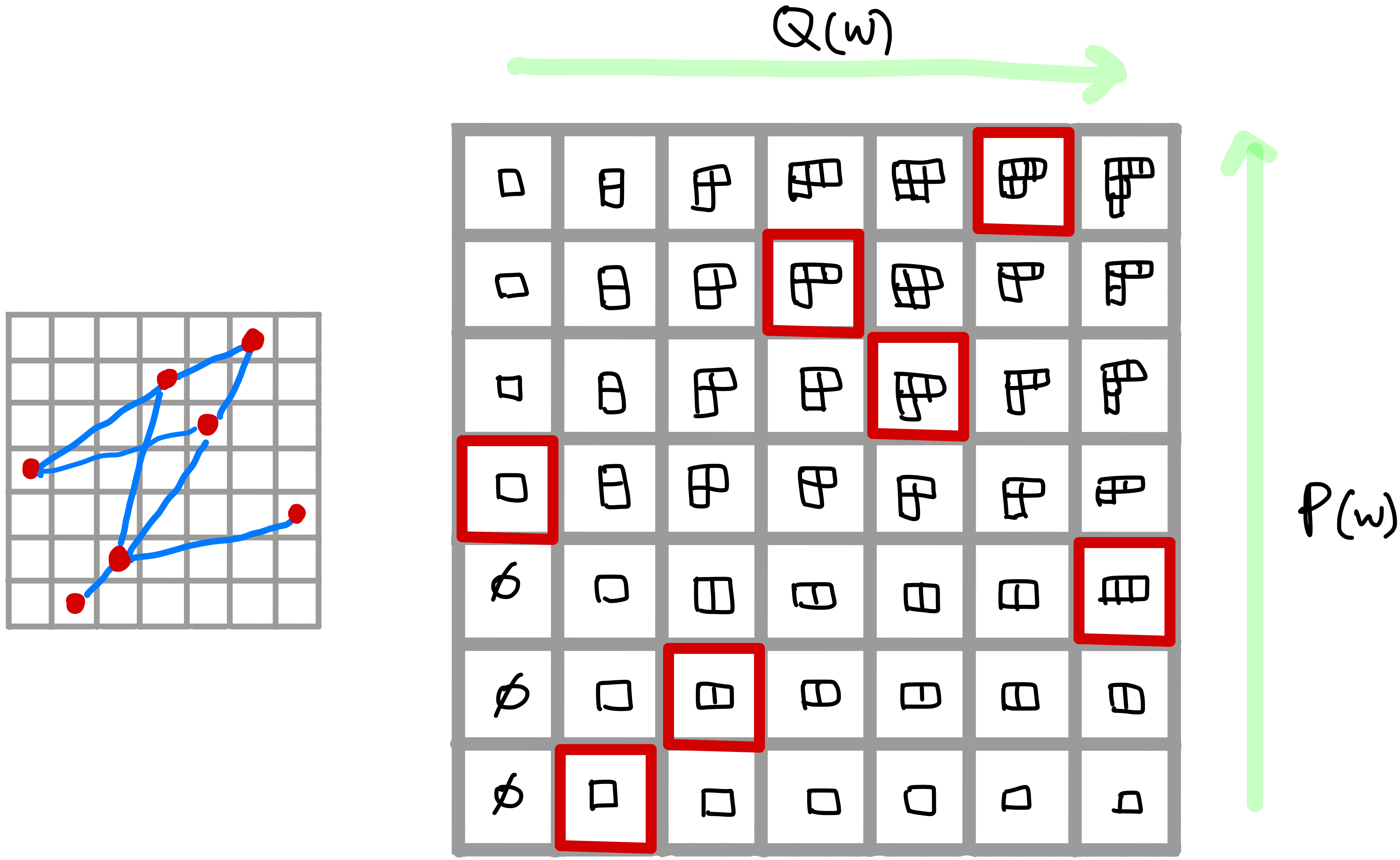
$\rightsquigarrow$ $P_w^{(5,5)} =$



$\rightsquigarrow$ $\lambda_{5,5} =$



Get an $n\times n$ array called the growth diagram (Fomin).


Q(w)
P(w)

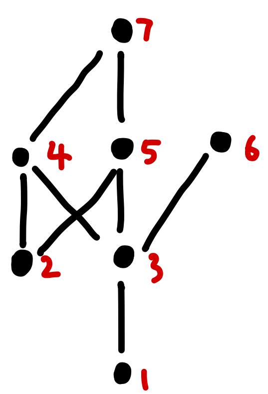
§ 3 Jordan Blocks.

<u>Def</u> For every partial order P on $[n]$, its incidence algebra $I(P)$ contains $n \times n$ <u>nilpotent matrices</u> M s.t.

$$M_{i,j} = \text{generic non-zero} \quad \text{if } i < j$$

$$M_{i,j} = 0 \quad \text{if } i, j \text{ incomparable.}$$

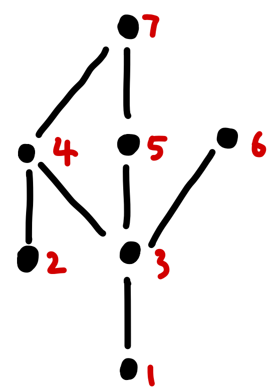
<u>E.g.</u>

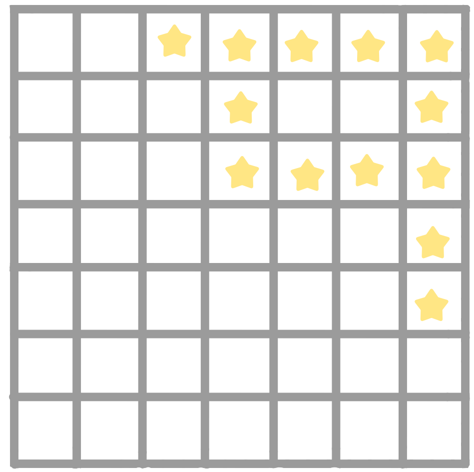


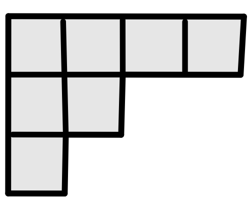
		★	★	★	★	★
			★	★		★
			★	★	★	★
						★
						★

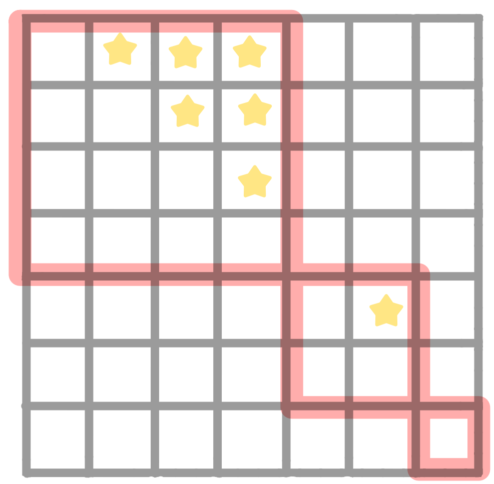
Thm (Gansner 1981 , Saks 1980)

The Jordan blocks of any $M \in I(P)$ is determined by $\lambda(P)$









§ Complete Flag Varieties

A (complete) flag is $\phi = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$.

The flag variety $Fl_n(\mathbb{C})$ is the alg. variety contains all such flags.

$$\text{"} Fl_n(\mathbb{C}) \cong GL_n(\mathbb{C})/B \text{"}$$

Fix a basis $\{e_1\ e_2 \cdots e_n\}$, the standard flag E^{id} is

$$E_\bullet^{id} = \phi \subset \langle e_1 \rangle \subset \langle e_1 e_2 \rangle \subset \cdots \subset \mathbb{C}^n$$

For $w \in S_n$, the permutation flag E^w is

$$E_\bullet^w = \phi \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)} e_{w(2)} \rangle \subset \cdots \subset \mathbb{C}^n$$

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$$\langle e_2 \rangle \subset \langle e_2 e_3 \rangle \subset \langle e_2 e_3 e_1 \rangle$$

Relative Position

For a pair of flags $E_\bullet : 0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n$
$F_\bullet : 0 \subset F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$,

their relative position $d(E,F)$ is the matrix $\underset{n \times n}{D}$ where

$$D_{ij} = \dim(E_i \cap F_j)$$

this is used to define Schubert cells:

$$X_o^w = \{ F_\bullet \subset Fl_n(\mathbb{C}) : d(E_\bullet^{id}, F_\bullet) = d(E_\bullet^{id}, E_\bullet^{w}) \} = \underline{B} w B / B$$

Fact $d(E,F)$ is the north-west rank matrix of some permutation, thus can identify $d(E,F)$ with that permutation.

E.g.: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightsquigarrow 312$

Alternatively, the relative position of E & F is the permutation w such that if $E \cong \{0 \subset \{v_1\} \subset \{v_1, v_2\} \subset \cdots \subset \mathbb{C}^n\}$

then $F \cong \{0 \subset \{v_{w(1)}\} \subset \{v_{w(1)}, v_{w(2)}\} \subset \cdots \subset \mathbb{C}^n\}$

Get combinatorial data from flags:

Def We say a nilpotent x contracts a flag $F_\bullet$ if

$$x F_{i+1} \subset F_i \quad \forall i$$

★ For a nilpotent x, can get a Young diagram $\lambda(x)$ from its Jordan blocks.

★ Moreover, by restricting x on components of $F_\bullet$, can get a sequence of Young diagrams:

$$\emptyset \subseteq \lambda(x|_{F_1}) \subseteq \lambda(x|_{F_2}) \subseteq \cdots \subseteq \lambda(x)$$

i.e. a Young tableau!

Note. The $\subseteq$'s are strict $\subsetneq$ when x is <u>generic</u>, in which case the tableau is a SYT.

Relative Position via RS (steinberg)

Take two flags $E_\bullet$ & $F_\bullet$, let x be a generic nilpotent which contracts both $E_\bullet$ and $F_\bullet$.

Get two SYTs:

$$T_E : \emptyset \subset \lambda(x|_{E_1}) \subset \lambda(x|_{E_2}) \subset \cdots \subset \lambda(x)$$

$$T_F : \emptyset \subset \lambda(x|_{F_1}) \subset \lambda(x|_{F_2}) \subset \cdots \subset \lambda(x)$$

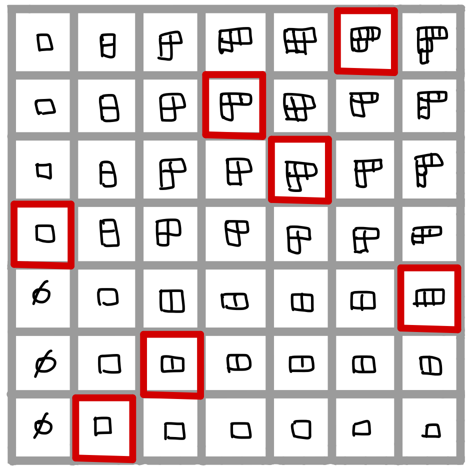
then the RS-correspondence tells us ...

$$(T_E, T_F) \xleftrightarrow{RS} w \boxed{= d(E_\bullet, F_\bullet)}$$

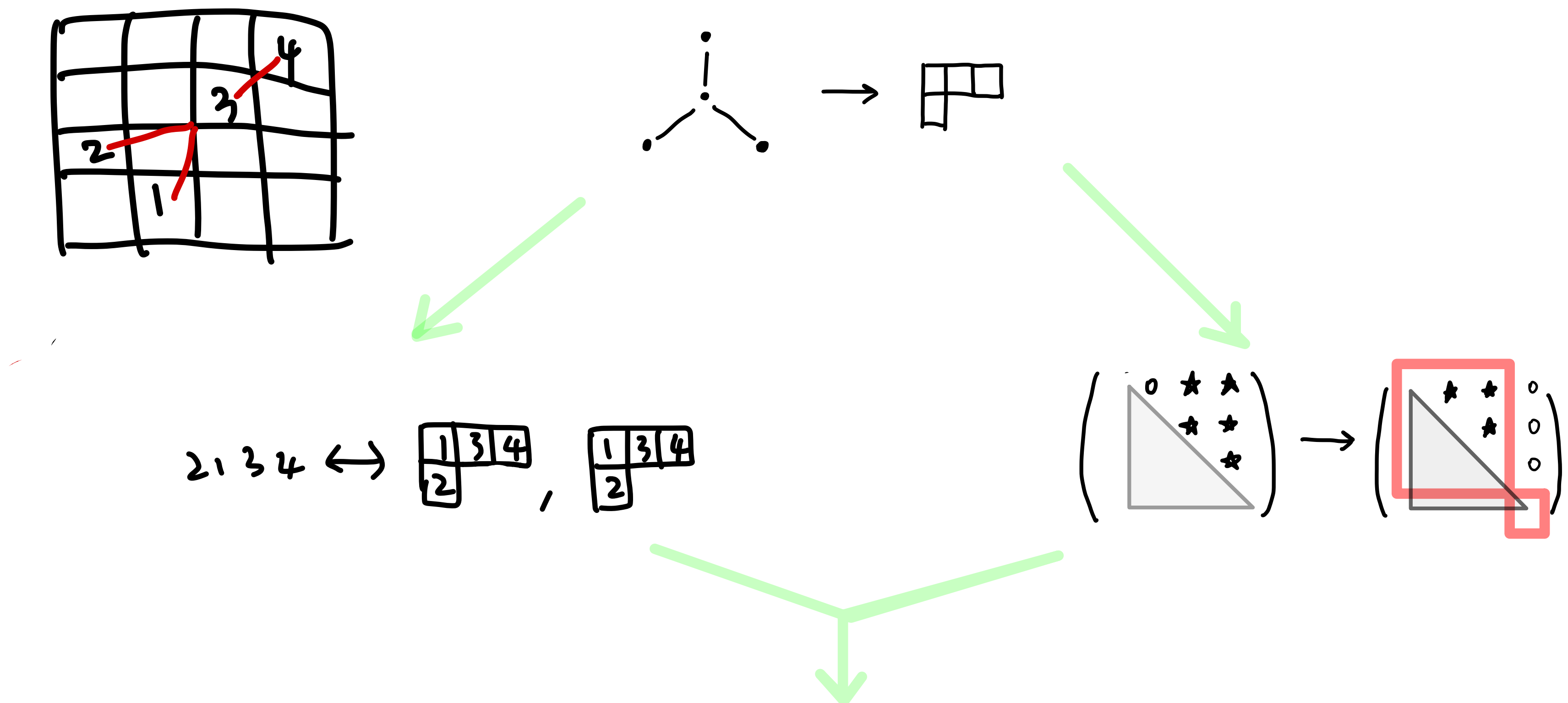
the relative position of $E_\bullet$ & $F_\bullet$

... Why?

Let $\lambda_{ij} = \lambda(x|_{E_i \cap E_j})$, then the $n \times n$ array $\{\lambda_{ij}\}$ is the same as the "growth diagram".



Pf idea Given $\lambda(x|_{E_{i-1} \cap F_j})$, $\lambda(x|_{E_{i-1} \cap F_{j-1}})$, $\lambda(x|_{E_i \cap F_{j-1}})$, find $\lambda(x|_{E_i \cap F_j})$ by case-by-case analysis of the Jordan chains. The rules agree with the recursive calculation for GK-shape (Thm 2).



$E: \emptyset \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1 e_2 e_3 \rangle \subset \mathbb{C}^4$

$F: \emptyset \subset \langle e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1 e_2 e_3 \rangle \subset \mathbb{C}^4$

?

Thank You for Listening!